

Effect of a Bias Field on Disordered Wave Guides: Universal Scaling of Conductance and Application to Ultracold Atoms

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We study the transmission and conductance of a disordered matter wave guide subjected to a finite bias force field. We show that the statistical distribution of transmission takes a universal form. This distribution depends on a single parameter, the system length expressed in a rescaled metrics, which encapsulates all the microscopic features of the medium. Excellent agreement with numerics is found for various models of disorder and bias. For white-noise disorder and a constant force, we find algebraic decay of the transmission with distance, irrespective of the value of the force. The observability of these effects in ultracold atomic gases is discussed, taking into account specific features, such as finite-range disorder correlations, inhomogeneous forces, and finite temperatures.

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Introduction.— Anderson localization in homogeneous disordered materials is traditionally signaled by exponential suppression of diffusion and conductance [1–3]. The connection between the two is firmly established by linear-response theory and the Einstein-Sutherland relation [4, 5]. Hence, the decay of localized wave packets and transmission coefficients with distance are characterized by the same localization length. Bias fields, such as electric forces, induce strong nonlinear response, which significantly affect localization and question this relation. For a weak bias field, algebraic (rather than exponential) localization of the wave-functions has been established by previous numerical [6] and analytical [7] work. Furthermore, Ref. [7] presented a rigorous proof that states become extended beyond a critical value of the dimensionless parameter $\alpha = \hbar^2 F / m U_R$ (with F the force, m the particle mass, and U_R the disorder strength), see also Ref. [8]. This is qualitatively consistent with a diagrammatic calculation of the asymptotic density of an expanding wave packet [9], yielding $n(x) \sim 1/x^{\beta_{\text{dens}}}$ in the direction of the force, with $\beta_{\text{dens}} = 1 + (1 - \alpha)^2 / 8\alpha$ for $\alpha < 1$. For $\alpha > 1$, the asymptotic density is not normalizable, hence signaling a delocalization transition at $\alpha = 1$. In contrast, numerical evidence was provided in Ref. [6] that the transmission coefficient remains algebraic for arbitrary large values of the force, $\exp(\overline{\ln T}) \sim 1/x^{\beta_{\text{tr}}}$, and is thus unaffected by the delocalization transition. Moreover the exponent $\beta_{\text{tr}} \simeq 1/2\alpha$ found numerically significantly differs from the exponent β_{dens} . While striking, it is worth noting that this result was found for the specific Kronig-Penney lattice model and by averaging the logarithm of the transmission rather than the transmission itself. Therefore the behavior of physical quanti-

ties that are directly related to the transmission, such as the Landauer conductance, remains unclear, in particular for generic models of disorder. This question has direct applications in mesoscopic physics to understand the electric response to a bias field in disordered carbon nanotubes [10] or silicon nanowires [11] for instance. It may have even greater relevance to ultracold atoms where Anderson localization has been extensively studied [12–22]. In those systems, a tunable bias field can easily be applied, and the conductance is now accessible via the discharge between well-controlled reservoirs [23–25].

Here we study the transmission of a one-dimensional disordered matter wave guide subjected to arbitrary disorder and bias force field. For white-noise disorder and a constant force, we find algebraic decay of the transmission coefficient for arbitrary force strength and derive the exponent $\beta_{\text{tr}} = 1/2\alpha$ analytically, hence generalizing the result of Ref. [6] to generic white-noise disorder. Moreover, we derive a universal form of the statistical distribution of the transmission coefficient. It is characterized by a unique parameter, the length of the wave guide expressed in a rescaled metrics, which encapsulates all microscopic features of the medium. We also perform numerical calculations for various models, and obtain excellent agreement with analytical calculations. Application to Landauer conductance measurements in ultracold-atom experiments is discussed. In particular, we consider the effect of finite disorder correlations, inhomogeneous forces, and finite temperatures.

Statistical distribution of transmission.— To start with, consider the transmission of a matter wave of energy $E \geq 0$ in a disordered material of length L in the presence of a bias force field $F(x)$ (see dashed black rect-

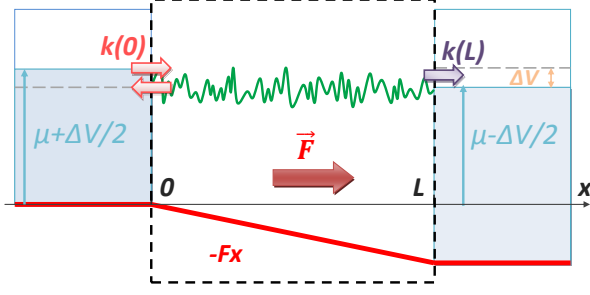


Figure 1. Transmission and conductance of a matter wave guide of length L (black rectangle) in the presence of a bias force $F(x)$ and disorder (random green line). The bias potential is shown for a constant force (red line). The incident and reflected wave vectors are $k(0)$ and the transmitted one is $k(L)$. The Landauer conductance is measured from the discharge between two reservoirs (left and right blue boxes) with average chemical potential μ and infinitesimal potential difference ΔV .

angle in Fig. 1). In the following, we will use the semi-classical kinetic energy $K(x) \equiv E + \int_0^x dx' F(x')$ as well as its associated wave vector $k(x) = \sqrt{2mK(x)}/\hbar$ and wavelength $\lambda(x) = 2\pi/k(x)$. We assume that the force $F(x)$ has no strong negative values, so that $K(x)$ is positive everywhere. The disordered potential $V(x)$ is homogeneous and Gaussian. Its average is zero and its two-point correlation function reads $C(x) \equiv \overline{V(x')V(x'+x)}$. The latter may model white-noise disorder, $C(x) = U_R \delta(x)$ with U_R the disorder strength, or correlated disorder, $C(x) = (U_R/\sigma_R) \times c(x/\sigma_R)$ with σ_R the correlation length and the normalization $\int du c(u) = 1$.

To compute the statistical distribution of the transmission coefficient, we use the transfer matrix approach. We briefly outline its generalization to include a, possibly inhomogeneous, bias force field (see details in the Supplemental Material). Assuming that the (non-averaged) transmission, $T(x)$, and reflection, $R(x) = 1 - T(x)$, coefficients from the origin to the distance x are known, we add an infinitesimal cell of material of length Δx , with transmission and reflection coefficients $T_{\Delta x}(x)$ and $R_{\Delta x}(x)$, respectively. The transmission coefficient at length $x + \Delta x$ is then computed from the product of the two transfer matrices. It yields

$$T(x + \Delta x) = \frac{T(x)T_{\Delta x}(x)}{|1 - \sqrt{R(x)R_{\Delta x}(x)} e^{i\theta_{\Delta x}(x)}|^2}, \quad (1)$$

where $\theta_{\Delta x}(x)$ is the phase accumulated during one total internal reflection at point x . For weak disorder, i.e. $\ell_-(x) \gg \lambda(x), \sigma_R$ with $\ell_-(x)$ the back-scattering mean free path at kinetic energy $K(x)$, we may choose intermediate elementary lengths Δx , such that $\lambda(x), \sigma_R \ll \Delta x \ll \ell_-(x)$. Since $\Delta x \ll \ell_-(x)$, the non-vanishing value of the reflection coefficient $R_{\Delta x}(x)$ results from typ-

ically less than one scattering and it may thus be computed in the single-scattering approximation. Assuming that the work of the force on the length of the elementary cell is small, $F(x)\Delta x \ll K(x)$, $\hbar^2 k(x)/2m\partial_k \ln \tilde{C}[2k(x)]$, we find $R_{\Delta x}(x) \simeq \Delta x/\ell_-(x) \ll 1$ with

$$\ell_-(x) \simeq 2\hbar^2 K(x)/m\tilde{C}[2k(x)], \quad (2)$$

where $\tilde{C}[2k(x)]$ is the disorder power spectrum. We may then develop the right-hand-side of Eq. (1) in powers of $R_{\Delta x}(x)$. It yields a Markov process for the variable $T(x)$. The Kramers-Moyal expansion of the master equation for the distribution of transmission at point x , $P(T, x)$, reduces to its first two moments under the random-phase approximation on $\theta_{\Delta x}(x)$. We thus find the Fokker-Planck equation

$$\ell_-(x) \frac{\partial P}{\partial x} = \frac{\partial T^2 P}{\partial T} + \frac{\partial^2}{\partial T^2} (T^2(1-T)P), \quad (3)$$

with the initial condition $P(T, x=0) = \delta(T-1)$. Note that, for a positive bias force field F and a bounded disorder power spectrum \tilde{C} , the back-scattering mean free path $\ell_-(x)$ increases and the wave length $\lambda(x)$ decreases with the distance x , so that all the validity conditions of Eq. (3) are always fulfilled, at least in the asymptotic limit $x \rightarrow \infty$.

It follows from Eq. (3) that the quantity $\ell_-(x)$ provides the natural metrics in the disordered material in the presence of the bias field. Using the inhomogeneous dimensionless coordinate

$$s(x) = \int_0^x \frac{dx'}{\ell_-(x')}, \quad (4)$$

the quantity $\ell_-(x)$ disappears from Eq. (3) [26]. The resulting differential equation for $P(T, L)$ admits the analytic solution [27]

$$P(T, L) = \frac{2e^{-s(L)/4}}{\sqrt{\pi}s(L)^{3/2}T^2} \int_{\cosh^{-1} \sqrt{1/T}}^{\infty} dy \frac{y e^{-y^2/s(L)}}{\sqrt{\cosh^2 y - 1/T}}, \quad (5)$$

which generalizes the celebrated log-normal law. The main result of this approach is that the solution (5) is universal. All the microscopic features of the medium, such as disorder correlations and bias field, are fully encapsulated into the definition of the metrics, Eq. (4).

Algebraic localization.— We now compute characteristic quantities and compare them to exact numerical calculations. Let us start with the logarithm of the transmission. We find $\ln T(L) = -s(L)$, which calls for several remarks. First, this formula exactly matches the heuristic formula proposed in Ref. [6] to interpret numerical results found for the specific Kronig-Penney model. Our analysis justifies this formula on rigorous grounds and generalizes it to any model of disorder and bias field. Second, for white-noise disorder and constant force, where

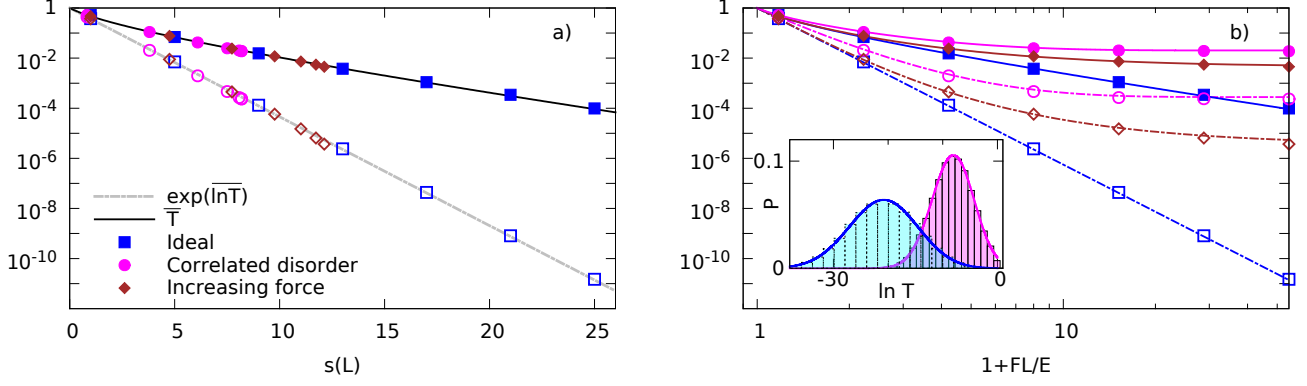


Figure 2. Analytical prediction versus numerical results for \overline{T} (solid black line and filled symbols) and $\exp(\overline{\ln T})$ (dashed-dotted gray line and open symbols), for $\alpha = 0.08$ and $2mF/\hbar^2 k_{in}^3 = 0.01$ in the following cases: constant force and white-noise disorder (blue squares), constant force and Gaussian correlated disorder with $\sigma_R k(0) = 0.3$ (magenta circles), harmonic force with $2m\omega/\hbar k(0)^2 = 0.01$ and uncorrelated disorder (brown diamonds). (a) Results plotted as a function of the metrics $s(L)$ [for white noise and constant force, $s(L) \propto \ln(1 + FL/E)$, see Eq. (6)]. (b) Results plotted as a function of $1 + FL/E$. Inset: Probability distribution of $\ln T$ for $FL/E = 28$, as found from numerical simulations (histogram) compared to analytical prediction (line), for a constant force, with correlated (dark, magenta) and uncorrelated (light, blue) disorder.

$\tilde{C}(2k) = U_R$ and $K(x) = E + Fx$, we find

$$s(L) = \int_0^L dx \frac{mU_R}{2\hbar^2(E + Fx)} = \frac{1}{2\alpha} \ln \left(1 + \frac{FL}{E} \right), \quad (6)$$

where $\alpha = \hbar^2 F/mU_R$ is the relative strength of the force and the disorder [9]. It yields the characteristic algebraic decay $\exp(\overline{\ln T}) \sim 1/L^{1/2\alpha}$. Excellent agreement with exact numerical calculations is found for continuous white-noise disorder [see the open blue squares and dotted-dashed line on Fig. 2(a)]. Third, the relative fluctuations of the logarithmic transmission equal 1 at small $s(L)$ and decrease as $\Delta \ln T(L)/|\ln T(L)| = \sqrt{2/s(L)}$ for large values of $s(L)$. Hence, the quantity $\ln T$ is self-averaging for white-noise disorder and constant force since the rescaled length $s(L)$ then diverges in the long-distance limit.

We now turn to the average transmission, which is more directly related to physically-relevant quantities (see below). Using Eq. (5), we find the exact formula

$$\overline{T(L)} = \frac{4e^{-s(L)/4}}{\sqrt{\pi}s(L)^{3/2}} \int_0^\infty dy \frac{y^2 e^{-y^2/s(L)}}{\cosh(y)}. \quad (7)$$

Again, it is in excellent agreement with exact numerical calculations [see filled blue squares and solid line on Fig. 2(a)]. Note that the relative fluctuations of T rapidly increase for large $s(L)$, $\Delta T/\overline{T} \sim s(L)^{3/4} e^{s(L)/8}$, so that the numerical calculation of \overline{T} requires a huge number of realizations of the disorder. For white-noise disorder and constant force, we find the asymptotic behavior $\overline{T(L)} \sim 1/L^{1/8\alpha}$, up to logarithmic corrections. The difference of the scalings $\exp(\overline{\ln T(L)}) \sim 1/L^{1/2\alpha}$ and $\overline{T(L)} \sim 1/L^{1/8\alpha}$ is not surprising. It is reminiscent of

the large fluctuations associated to the log-normal law as well known in the absence of a bias field [28].

In turn, it is quite surprising that the scaling $\overline{T(L)} \sim 1/L^{1/8\alpha}$ differs from that found for the density profile of an expanding wave packet, $\overline{n(x)} \simeq 1/x^{1+(1-\alpha)^2/8\alpha}$ [9]. To understand this difference, consider a particle initially at position $x = 0$ and look at the probability that it has been transmitted beyond $x = L$ after infinite time. If the disorder is restricted to the space interval $[0, L]$, this probability is given by the transmission coefficient $\overline{T(L)}$ and therefore decays as $L^{-1/8\alpha}$. In contrast, if the disorder extends over the full x line, it turns into $\int_L^\infty dx \overline{n(x)}$, which decays as $L^{-(1-\alpha)^2/8\alpha}$. This slower decay may be ascribed to the presence of long-range algebraically-localized eigenstates, centered beyond $x = L$, whose overlap with the initial wavefunction is significant, thus enhancing the probability of finding the particle at $x > L$. Note that this effect is expected to be less important when the eigenstates are more strongly localized, i.e. when α vanishes. This is consistent with the equality of the two exponents, $(1-\alpha)^2/8\alpha \simeq 1/8\alpha$ in the limit $\alpha \rightarrow 0$.

Landauer conductance.— The statistical distribution of transmission determines a number of physical quantities, and in the first place Landauer conductance [29, 30]. The latter is defined as the ratio of the current I induced by the potential imbalance between two charge reservoirs to their potential difference ΔV , at zero temperature. In the limit $\Delta V \rightarrow 0$, the disorder-average conductance reads $\overline{G_n(\mu)} = \int dT P(T, L) f_n[T(\mu)]$, where $\mu \pm \Delta V/2$ are the chemical potentials of the reservoirs (see Fig. 1), $P(T, L)$ is given by Eq. (5), and the function $f_n(T)$ depends on the specific scheme [30]. Here we focus on the simplest case where ΔV is measured inside the reservoirs

(two-terminal scheme). One then finds the conductance $\overline{G}_2(\mu) = G_0 \times \overline{T}(\mu)$ with G_0 the conductance quantum. The quantity \overline{G}_2 may be measured in mesoscopic [31] or ultracold-atom [23–25] systems for instance. Extension to multi-terminal configurations is straightforward.

To discuss the behavior of G_2 , or equivalently T , it is worth including specific features beyond the ideal case considered so far, in particular disorder correlations. Using the Gaussian correlation function $C(x) = \frac{U_R}{\sigma_R \sqrt{2\pi}} \exp(-x^2/2\sigma_R^2)$, where σ_R is the correlation length, we find that the algebraic decay is clearly visible at short distance but is strongly suppressed in large-distance limit [see numerics on Fig. 2(b), magenta filled (\overline{T}) and open ($\ln \overline{T}$) circles]. This entails correlation-induced delocalization. This behavior is easily understood from that of the metrics $s(L)$, which now reads

$$s(L) = \int_0^L dx \frac{m\tilde{C}[2k(x)]}{2\hbar^2(E + Fx)}, \quad (8)$$

where $\tilde{C}[2k(x)] = U_R \exp\left(-\frac{E+Fx}{\hbar^2/4m\sigma_R^2}\right)$. Compared to white-noise disorder, here the linear increase of the kinetic energy $K(x) = E + Fx$, which appears in the argument of the power spectrum \tilde{C} , suppresses the logarithmic divergence of $s(L)$. Hence the metrics $s(L)$, the distribution $P(T, L)$, and all disorder-average functions of T saturate when $L \rightarrow \infty$. This delocalization effect is not specific to Gaussian correlations but applies to any model of disorder with finite-range correlations. Similarly, faster-than-linear bias also entails delocalization since it makes $s(L)$ converge. This is confirmed by numerical calculations performed with the slightly linearly increasing force $F(x) = F + m\omega^2 x$ [see Fig. 2(b), brown diamonds].

Correlation-induced delocalization effects appear for $\Delta k(x)\sigma_R \sim 1$, with $\Delta k(x) = k(x) - k(0)$ and σ_R is the disorder correlation length. In the two-terminal configuration of Fig. 1, $k(0) = k_F = \sqrt{2mk_B\theta_F}/\hbar$ is the Fermi wavevector of the left-hand-side reservoir, with θ_F the Fermi temperature and k_B the Boltzmann constant. Using $\Delta k(x) \sim \partial_x k(0) \times x \sim mFx/\hbar^2 k_F$, we find $x \sim \sqrt{\frac{2k_B\theta_F}{m\sigma_R^2}} \frac{\hbar}{F}$. For mesoscopic channels designed in ultracold-atom systems [23, 25], where typically $\theta_F \sim 500\text{nK}$ and $\sigma_R \sim 0.5\mu\text{m}$, and assuming that the force results from gravity on ^6Li atoms, $F \simeq 10^{-25}\text{N}$, we find $x \sim 80\mu\text{m}$. Since this value is of the order of magnitude of the channel length, correlation-induced delocalization effects can be significant in these systems. To circumvent this issue, one can take advantage of the universality of the distribution of transmission (5). Indeed, as shown on Fig. 2(a), we recover a universal behavior of the transmission by rescaling the Euclidean distance L to the metrics $s(L)$ for correlated disorder, as well as non-constant force [32].

Let us finally consider the effect of a finite temperature, which is typically $\theta \sim 0.1 - 0.3\theta_F$ in ultracold-atom

systems [23, 33]. Using the Sommerfeld expansion of the current-potential characteristic function, we find

$$\overline{G}_2(L) = G_0 \left\{ \overline{T}[s(L)] + \frac{\pi^2(k_B\theta)^2}{6} A(L) \right\} \quad (9)$$

where, for a constant force F ,

$$A(L) = \frac{1}{F} \frac{\partial}{\partial \mu} \left[\frac{1}{\ell_-(L)} - \frac{1}{\ell_-(0)} \right] \times \frac{\partial \overline{T}}{\partial s} + \frac{1}{F^2} \left[\frac{1}{\ell_-(L)} - \frac{1}{\ell_-(0)} \right]^2 \times \frac{\partial^2 \overline{T}}{\partial s^2}. \quad (10)$$

For the parameters above and the typical disorder strength $V_R/k_B \sim 0.5\mu\text{K}$, we find that finite-temperature effects contribute the conductance \overline{G}_2 from less than 2% for $\theta \sim 0.1\theta_F$ up to 15% for $\theta \sim 0.3\theta_F$. Since the quantity $A(L)$ is not a universal function of the rescaled length $s(L)$, such finite-temperature effects break universal scaling. However, if the system is sufficiently large so that $\ell_-(L) \gg \ell_-(0)$, universal scaling is recovered.

Outlook.— In conclusion, we have computed the statistical distribution of transmission in a disordered matter wave guide in the presence of a bias force. For white-noise disorder and a constant force, we have shown that the transmission decays algebraically, irrespective of the value of the force, in agreement with numerical calculations [6]. This behavior differs from the expansion of a wave packet, which features a delocalization transition [9] (see also Refs. [7, 8]). These different behaviors have been traced back to the long-range character of the algebraic decay of the localized eigenstates in the presence of the bias. They could be directly observed in ultracold-atom experiments, which allow for both transmission and expansion schemes [15, 16, 25]. While finite-range disorder correlations or non-constant forces may entail saturation of the transmission, we have shown that a universal behavior can be recovered using appropriate rescaling. Finite-temperature effects have also been discussed.

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- [1] P. W. Anderson, “Absence of diffusion in certain random lattices,” *Phys. Rev.* **109**, 1492 (1958).
 - [2] P. A. Lee and T. V. Ramakrishnan, “Disordered electronic systems,” *Rev. Mod. Phys.* **57**, 287 (1985).
 - [3] E. Abrahams, *50 Years of Anderson Localization* (World Scientific, Singapore, 2010).

- [4] J. Rammer, *Quantum Transport Theory* (Perseus Books, Reading, Massachusetts, 1998).
- [5] E. Akkermans and G. Montambaux, *Mesoscopic Physics of Electrons and Photons* (Cambridge University Press, Cambridge, 2007).
- [6] C. M. Soukoulis, J. V. José, E. N. Economou, and P. Sheng, “Localization in one-dimensional disordered systems in the presence of an electric field,” *Phys. Rev. Lett.* **50**, 764 (1983).
- [7] F. Delyon, B. Simon, and B. Souillard, “From power-localized to extended states in a class of one-dimensional disordered systems,” *Phys. Rev. Lett.* **52**, 2187 (1984).
- [8] F. Bentosela, V. Grecchi, and F. Zironi, “Stark-Wannier states in disordered systems,” *Phys. Rev. B* **31**, 6909 (1985).
- [9] V. N. Prigodin, “One-dimensional disordered system in an electric field,” *Zh. Eksp. Teor. Fiz.* **79**, 2338 (1980), [*Sov. Phys. JETP* **52**, 1185 (1980)].
- [10] S. Hong and S. Myung, “A flexible approach to mobility,” *Nat. Nanotechnol.* **2**, 207 (2007).
- [11] M. Hasan, M. F. Huq, and Z. H. Mahmood, “A review on electronic and optical properties of silicon nanowire and its different growth techniques,” *SpringerPlus* **2**, 1 (2013).
- [12] L. Sanchez-Palencia, D. Clément, P. Lugan, P. Bouyer, G. V. Shlyapnikov, and A. Aspect, “Anderson localization of expanding Bose-Einstein condensates in random potentials,” *Phys. Rev. Lett.* **98**, 210401 (2007).
- [13] B. Shapiro, “Expansion of a Bose-Einstein condensate in the presence of disorder,” *Phys. Rev. Lett.* **99**, 060602 (2007).
- [14] R. C. Kuhn, O. Sigwarth, C. Miniatura, D. Delande, and C. A. Müller, “Coherent matter wave transport in speckle potentials,” *New J. Phys.* **9**, 161 (2007).
- [15] J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D. Clément, L. Sanchez-Palencia, P. Bouyer, and A. Aspect, “Direct observation of Anderson localization of matter waves in a controlled disorder,” *Nature (London)* **453**, 891 (2008).
- [16] G. Roati, C. D’Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, “Anderson localization of a non-interacting Bose-Einstein condensate,” *Nature (London)* **453**, 895 (2008).
- [17] M. Piraud, P. Lugan, P. Bouyer, A. Aspect, and L. Sanchez-Palencia, “Localization of a matter wave packet in a disordered potential,” *Phys. Rev. A* **83**, 031603 (2011).
- [18] S. S. Kondov, W. R. McGehee, J. J. Zirbel, and B. DeMarco, “Three-dimensional Anderson localization of ultracold matter,” *Science* **334**, 66 (2011).
- [19] F. Jendrzejewski, A. Bernard, K. Mueller, P. Cheinet, V. Josse, M. Piraud, L. Pezzé, L. Sanchez-Palencia, A. Aspect, and P. Bouyer, “Three-dimensional localization of ultracold atoms in an optical disordered potential,” *Nat. Phys.* **8**, 398 (2012).
- [20] G. Semeghini, M. Landini, P. Castilho, S. Roy, G. Spagnolli, A. Trenkwalder, M. Fattori, M. Inguscio, and G. Modugno, “Measurement of the mobility edge for 3D Anderson localization,” *Nat. Phys.* **11**, 554 (2015).
- [21] J. Chabé, G. Lemarié, B. Grémaud, D. Delande, P. Szriftgiser, and J. C. Garreau, “Experimental observation of the Anderson metal-insulator transition with atomic matter waves,” *Phys. Rev. Lett.* **101**, 255702 (2008).
- [22] M. Lopez, J.-F. Clément, P. Szriftgiser, J. C. Garreau, and D. Delande, “Experimental test of universality of the Anderson transition,” *Phys. Rev. Lett.* **108**, 095701 (2012).
- [23] J.-P. Brantut, J. Meineke, D. Stadler, S. Krinner, and T. Esslinger, “Conduction of ultracold fermions through a mesoscopic channel,” *Science* **337**, 1069 (2012).
- [24] D. Stadler, S. Krinner, J. Meineke, J.-P. Brantut, and T. Esslinger, “Observing the drop of resistance in the flow of a superfluid Fermi gas,” *Nature (London)* **491**, 736 (2012).
- [25] S. Krinner, D. Stadler, J. Meineke, J.-P. Brantut, and T. Esslinger, “Superfluidity with disorder in a thin film of quantum gas,” *Phys. Rev. Lett.* **110**, 100601 (2013).
- [26] For non-vanishing disorder, we find $0 < \ell_{-}(x) < \infty$, and the metrics is well defined.
- [27] A. A. Abrikosov, “The paradox with the static conductivity of a one-dimensional metal,” *Solid State Commun.* **37**, 997 (1981).
- [28] C. W. J. Beenakker, “Random-matrix theory of quantum transport,” *Rev. Mod. Phys.* **69**, 731 (1997).
- [29] R. Landauer, “Spatial variation of currents and fields due to localized scatterers in metallic conduction,” *IBM J. Res. Dev.* **1**, 223 (1957).
- [30] M. Buttiker, “Symmetry of electrical conduction,” *IBM J. Res. Dev.* **32**, 317 (1988).
- [31] R. De Picciotto, H. L. Stormer, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, “Four-terminal resistance of a ballistic quantum wire,” *Nature (London)* **411**, 51 (2001).
- [32] We have checked that the distribution of transmission, Eq. (5) still holds in the presence of finite correlation lengths and non linear bias, see inset of Fig. 2(b).
- [33] R. Jördens, L. Tarruell, D. Greif, T. Uehlinger, N. Strohmaier, H. Moritz, T. Esslinger, L. De Leo, C. Kollath, A. Georges, V. Scarola, L. Pollet, E. Burovski, E. Kozik, and M. Troyer, “Quantitative determination of temperature in the approach to magnetic order of ultracold fermions in an optical lattice,” *Phys. Rev. Lett.* **104**, 180401 (2010).

–Supplemental Material–

Effect of a Bias Field on Disordered Wave Guides: Universal Scaling of Conductance and Application to Ultracold Atoms

In this supplemental material, we provide details about the transfer matrix formalism for an inhomogeneous medium and the derivation of the Fokker-Planck equation [Eq. (3) of the main paper].

To compute the transmission coefficient of a particle submitted to a bias force field $F(x)$ through a disordered sample in the space interval $[0, L]$, it is convenient to define the semi-classical kinetic energy $K(x) \equiv E + \int_0^x dx' F(x')$ and the associated wave vector $k(x) = \sqrt{2mK(x)}/\hbar$. At any position x , we may write the particle wave function $\psi(x)$ and its derivative $\partial_x \psi(x)$ in the form

$$\begin{pmatrix} \psi(x) \\ \partial_x \psi(x) \end{pmatrix} = \begin{pmatrix} e^{ik(x)x} & e^{-ik(x)x} \\ ik(x)e^{ik(x)x} & -ik(x)e^{-ik(x)x} \end{pmatrix} \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix} \quad (\text{S1})$$

A unique solution $(\psi_+(x), \psi_-(x))$ exists provided the determinant of the above matrix does not vanish, i.e. $k(x) \neq 0$.

The particle flux, $j(x) \equiv \frac{\hbar}{2im} (\psi^* \partial_x \psi - \psi \partial_x \psi^*)$ then reads $j(x) = j_+(x) + j_-(x)$, where

$$j_{\pm}(x) = \pm \frac{\hbar k(x)}{m} |\psi_{\pm}(x)|^2 \quad (\text{S2})$$

are the right-moving (+) and left-moving (−) fluxes. The transmission coefficient is then defined as the ratio of right-moving fluxes at the boundaries of the sample in the case where the incident flux is right-moving, i.e. $j_-(L) = 0$, and reads

$$T(L) \equiv \frac{j_+(L)}{j_+(0)} = \frac{k(L)}{k(0)} \frac{|\psi_+(L)|^2}{|\psi_+(0)|^2}. \quad (\text{S3})$$

Under the same assumption of right-moving incident flux, the reflection coefficient is defined as the ratio of left-moving emergent flux and the right-moving incident flux, and reads

$$R(L) \equiv \frac{|j_-(0)|}{j_+(0)} = \frac{|\psi_-(0)|^2}{|\psi_+(0)|^2}. \quad (\text{S4})$$

Scattering matrix.— Consider now a finite sample in the interval $[x_1, x_2]$, where $0 \leq x_1 < x_2 \leq L$. We define the scattering matrix

$$\mathbf{S} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \quad (\text{S5})$$

such that

$$\begin{pmatrix} \sqrt{k(x_1)}\psi_-(x_1) \\ \sqrt{k(x_2)}\psi_+(x_2) \end{pmatrix} = \mathbf{S} \begin{pmatrix} \sqrt{k(x_1)}\psi_+(x_1) \\ \sqrt{k(x_2)}\psi_-(x_2) \end{pmatrix}. \quad (\text{S6})$$

Particle-flux conservation between x_1 and x_2 imposes that the scattering matrix is unitary, $\mathbf{S}^\dagger \mathbf{S} = \mathbb{1}$. Moreover, time-reversal symmetry entails $\mathbf{S}^* = \mathbf{S}^\dagger$. Those two relations lead to the usual relations

$$\begin{cases} t = t' \\ |r|^2 + |t|^2 = |r'|^2 + |t|^2 = 1 \\ t^* r' + r^* t = 0. \end{cases} \quad (\text{S7})$$

Straightforward calculations then lead to the usual relations for the transmission and reflection coefficients of samples between the points x_1 and x_2 ,

$$T = |t|^2 \quad \text{and} \quad R = |r|^2 = |r'|^2. \quad (\text{S8})$$

Transfer matrix.— We now define the transfer matrix $\mathbf{T}(x_2, x_1)$ such that

$$\begin{pmatrix} \sqrt{k(x_2)}\psi_+(x_2) \\ \sqrt{k(x_2)}\psi_-(x_2) \end{pmatrix} = \mathbf{T}(x_2, x_1) \begin{pmatrix} \sqrt{k(x_1)}\psi_+(x_1) \\ \sqrt{k(x_1)}\psi_-(x_1) \end{pmatrix}. \quad (\text{S9})$$

Straightforward calculations yield

$$\mathbf{T} = \begin{pmatrix} 1/t^* & r'/t \\ -r/t & 1/t \end{pmatrix} \quad (\text{S10})$$

Transfer matrices can then be chained, i.e.

$$\mathbf{T}(x_n, x_1) = \mathbf{T}(x_n, x_{n-1})\mathbf{T}(x_{n-1}, x_{n-2})\dots\mathbf{T}(x_2, x_1). \quad (\text{S11})$$

Considering two samples in the intervals $[0, x]$ and $[x, x + \Delta x]$ respectively, where Δx is infinitesimal, one finds the relation

$$T(x + \Delta x) = \frac{T(x)T_{\Delta x}(x)}{|1 - \sqrt{R(x)R_{\Delta x}(x)}e^{i\theta_{\Delta x}(x)}|^2}, \quad (\text{S12})$$

where the sample $[0, x]$ has transmission coefficient $T(x)$ and reflection coefficient $R(x) = 1 - T(x)$, the sample $[x, x + \Delta x]$ has transmission coefficient $T_{\Delta x}(x)$ and reflection coefficient $R_{\Delta x}(x)$, and $r'(x)r_{\Delta x}(x) = |r'(x)r_{\Delta x}(x)|e^{i\theta_{\Delta x}(x)}$. For $\Delta x \ll \ell_-(x)$, we have $R_{\Delta x}(x) \ll 1$ and we may use the following expansion of $\Delta T(x) \equiv T(x + \Delta x) - T(x)$:

$$\Delta T(x) = T(x) \left\{ 2\sqrt{(1 - T(x))R_{\Delta x}(x)} \cos \theta_{\Delta x}(x) + R_{\Delta x}(x) [T(x) - 2 + 4(1 - T(x)) \cos^2 \theta_{\Delta x}(x)] \right\} + O(R_{\Delta x}(x)^{3/2}) \quad (\text{S13})$$

The transmission coefficient is thus governed by a stochastic process when the system length x increases. The Kramers-Moyal expansion of the corresponding master equation for the probability distribution of the transmission coefficient at a given length, $P(T, x)$, reads

$$\frac{\partial P(T, x)}{\partial x} = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial T^n} [M_n(T)P(T, x)] \quad (\text{S14})$$

with

$$M_n(T) = \left. \frac{\overline{(\Delta T(x))^n}}{\Delta x} \right|_{\Delta x \rightarrow 0}. \quad (\text{S15})$$

The overline denotes averaging over the disorder. Assuming that the quantity $\theta_{\Delta x}(x)$ is uniformly distributed on 2π and using $\overline{R_{\Delta x}(x)}$ is equal to $\Delta x/\ell_-(x)$, the average on both quantities can be performed independently. We then find

$$M_1 = -\frac{T^2(x)}{\ell_-(x)} \quad M_2 = \frac{2T^2(x)(1 - T(x))}{\ell_-(x)} \quad M_n = 0 \text{ for } n \geq 3. \quad (\text{S16})$$

The Kramers-Moyal expansion (S14) thus reduces to its first two moments, which yields the Fokker-Planck equation (3) of the main paper.